# PhDs in Logic
# Constructive Logic, Part II

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# Logical Laws

Constructive logic is logic without

LEM $\models A \vee \sim A$

and

DNE $\models \sim\sim A \supset A$.

# Logical Laws

Constructive logic is logic without

LEM $\vDash A \vee \sim A$

and

DNE $\vDash \sim\sim A \supset A$.

Really?

# Questions

What is a constructive logic?

What makes a logic constructive?

Might there be more than one constructive logics?

If so, which ones in particular?

If not, which is the right constructive logic?

# Outline

# Brouwer

# Example non-constructive proof

Does the equation $x^y = z$ have solutions in which $x$ and $y$ are both irrational, while $z$ is rational?

Consider the number$\sqrt{2}^{\sqrt{2}}$.

This number is either rational or it is not.

If it is, we're done.

If it isn't, take $x$ to be $\sqrt{2}^{\sqrt{2}}$ and set $y$ to $\sqrt{2}$.

Then $z$ is $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2.$ Done again.

Thus, whether or not $\sqrt{2}^{\sqrt{2}}$ is rational, we can answer our question affirmatively.

# Heyting and Kolmogorov

Heyting

Kolmogoroff

# Intuitionistic Logic: BHK-Interpretation

- $c$ is a proof of $A \wedge B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a proof of $A$ and $c2$ is a proof of $B$
- $c$ is a proof of $A \vee B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a proof of $A$ or $c2$ is a proof of $B$
- $c$ is a proof of $A \supset B$ iff $c$ is a construction that converts each proof of $A$ into a proof of $B$

# Negation

- nothing is a proof of $\perp$
- $c$ is a proof of $\sim A$ iff $c$ is a construction which transforms each proof of $A$ into a proof of $\perp$ .

$\perp$ is an absurd statement, such as 1=0.

To get full intuitionistic logic, we have to assume two things:

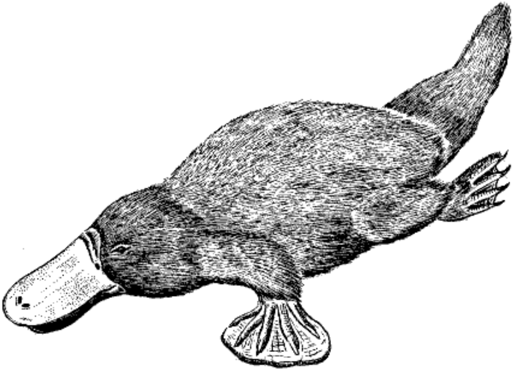
1. that a proof of any false statement can be transformed into a proof of 1=0
2. that a proof of 1=0 can be transformed into a proof of any statement

# Paraconsistency.

A logic is paraconsistent iff it does not validate the inference *ex contradictione quodlibet* (aka *Explosion*)

$A \wedge \neg A \models B$

# Minimal Logic



*The platypus of paraconsistent logics*

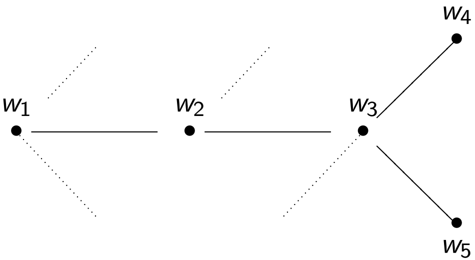
Introduced by I. Johanssen. In minimal logic, $A \wedge \neg A \vDash B$ does not hold, but $A \wedge \neg A \vDash \neg B$ does.

# Intuitionistic Logic: Kripke Semantics (I)

Model: $[W, \leq, v]$, $W$ is a set of worlds, $\leq$ is a partial order on those worlds, $v$ a valuation function that assigns 1 or 0 to atomic statements at all worlds.

Heredity: For each $p$: if $w \leq w$' and $v_w(p) = 1$ then $v_{w'}(p) = 1$

# Intuitionistic Logic: Kripke Semantics (II)




| | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|
| $p$ | 0 | 1 | 1 | 1 | 1 |
| $q$ | 0 | 0 | 0 | 1 | 1 |
| $r$ | 1 | 1 | 1 | 1 | 1 |
| $s$ | 0 | 0 | 0 | 1 | 0 |

1 means "proved", 0 means "not yet proved"

# Intuitionistic Logic: Kripke Semantics (III)

*For all* $w \in W$:
$w \Vdash_1 A \wedge B$ *iff* $w \Vdash_1 A$ *and* $w \Vdash_1 B$
$w \Vdash_1 A \vee B$ *iff* $w \Vdash_1 A$ *or* $w \Vdash_1 B$
$w \Vdash_1 A \supset B$ *iff for all* $x \geq w$, $x \Vdash_0 A$ *or* $x \Vdash_1 B$
$w \Vdash_1 \sim A$ *iff for all* $x \geq w$, $x \Vdash_0 A$

$\Gamma \vDash A$ *iff in every model and every* $w \in W$, *if* $w \vDash_1 B$ *for every* $B \in \Gamma$, *then* $w \Vdash_1 A$.

# Intuitionistic Logic vs. Minimal Logic

*For all* $w \in W$ *:*
$w \Vdash_1 A \wedge B$ *iff* $w \Vdash_1 A$ *and* $w \Vdash_1 B$
$w \Vdash_1 A \vee B$ *iff* $w \Vdash_1 A$ *or* $w \Vdash_1 B$
$w \Vdash_1 A \supset B$ *iff for all* $x \geq w$, $x \Vdash_0 A$ *or* $x \Vdash_1 B$
$w \Vdash_1 \bot$ *for no* $w$

$\sim A =_{def} A \supset \bot$

# Intuitionistic Logic vs. Minimal Logic

*For all* $w \in W$:

$w \Vdash_1 A \wedge B$ *iff* $w \Vdash_1 A$ *and* $w \Vdash_1 B$

$w \Vdash_1 A \vee B$ *iff* $w \Vdash_1 A$ *or* $w \Vdash_1 B$

$w \Vdash_1 A \supset B$ *iff for all* $x \geq w$, $x \Vdash_0 A$ *or* $x \Vdash_1 B$

$\sim A =_{def} A \supset \bot$

# Dummett

# Dummett's programme

*[The intuitionistic] theory of meaning generalizes readily to the non-mathematical case. Proof is the sole means which exists in mathematics for establishing a statement as true: the required general notion is, therefore, that of verification. On this account, an understanding of a statement consists in a capacity to recognize whatever is counted as verifying it, i.e. as conclusively establishing it as true. (...) The advantage of this conception is that the condition for a statement's being verified, unlike the condition for its truth under the assumption of bivalence, is one which we must be credited with the capacity for effectively recognizing when it obtains; hence there is no difficulty in stating what an implicit knowledge of such a condition consists in—once again, it is directly displayed by our linguistic practice. (WTM p.70)*

# Dummett's programme

Obtain a semantic account for empirical discourse by substituting "verification" for "proof" in the above.

# BHK - Interpretation for empirical discourse

$c$ is a verification of $A \wedge B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a verification of $A$ and $c2$ is a verification of $B$

$c$ is a verification of $A \vee B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a verification of $A$ or $c2$ is a verification of $B$

$c$ is a verification of $A \supset B$ iff $c$ is a procedure that converts each verification $d$ of $A$ into a verification $c(d)$ of $B$

nothing is a verification of $\perp$

# No empirical $\bot$

Dummett:

> *An explanation [of $\bot$] relies on the underlying presumption that, given a proof of a false numerical equation, we can construct a proof of any statement whatsoever.*
> *It is not obvious that, when we extend these conceptions to empirical statements, there exists any class of decidable atomic statements for which a similar presumption holds good; ...*

# Falsifications

*... and it is therefore not obvious that we have, for the general case, any similar uniform way of explaining negation for arbitrary statements.*
*It would therefore remain well within the spirit of a theory of meaning of this type that we should regard the meaning of each statement as being given by the simultaneous provision of a means for recognizing a verification of it and a means for recognizing a falsification of it (...).*

# Conditionals

Another reason for Dummett to turn to falsifications: Indicative Conditionals

"If $A$, then $B$" is verified iff ... ?

"If $A$, then $B$" is falsified iff $A$ is verified and $B$ is falsified.

# How to fix negation with falsifications

$c$ is a verfication of $-A$ iff $c$ is a falsification of $A$

$c$ is a falsification of $-A$ iff $c$ is a verification of $A$.

This approach will lead us to Nelson logic aka $N_3$.

# Alternative Account of Negation: Nelson Logic

Wansing

# Lopez-Escobar Interpretation

$c$ is a verification of $A \wedge B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a verification of $A$ and $c2$ is a verification of $B$

$c$ is a falsification of $A \wedge B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a falsification of $A$ or $c2$ is a falsification of $B$

$c$ is a verification of $A \vee B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a verification of $A$ or $c2$ is a verification of $B$

$c$ is a falsification of $A \vee B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a falsification of $A$ and $c2$ is a falsification of $B$

# Lopez-Escobar Interpretation

$c$ is a verification of $A \supset B$ iff $c$ is a procedure that converts each verification $d$ of $A$ into a verification $c(d)$ of $B$

$c$ is a falsification of $A \supset B$ iff $c$ is a pair $(c1, c2)$ such that $c1$ is a verification of $A$ and $c2$ is a falsification of $B$

$c$ is a verification of $-A$ iff $c$ is a falsification of $A$

$c$ is a falsification of $-A$ iff $c$ is a verification of $A$.

# Kripke Semantics N3 I

Model: $[W, \leq, v]$, $W$ worlds, $\leq$ partial order, $v$ a partial function from formulas to 1 and 0.

Heredities:

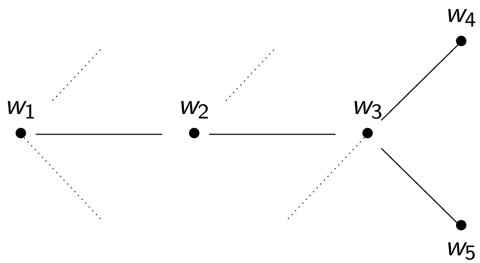
For all $p$, and all worlds $w$ and $w'$, if $w \leq w'$ and $w \Vdash_1 p$, then $w' \Vdash_1 p$, and

for all $p$, and all worlds $w$ and $w'$, if $w \leq w'$ and $w \Vdash_0 p$, then $w' \Vdash_0 p$.

Consequence $N_3$:

> $\Gamma \vDash A$ *iff in every model and every* $w \in W$, *if* $w \Vdash_1 B$ *for every* $B \in \Gamma$, *then* $w \Vdash A$.

# Kripke Semantics N3 II




| | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|
| $p$ | - | - | 1 | 1 | 1 |
| $q$ | - | 0 | 0 | 0 | 0 |
| $r$ | 1 | 1 | 1 | 1 | 1 |
| $s$ | - | - | - | 0 | 0 |

Value 1 means "verified", value 0 "falsified".

# Kripke Semantics N3 III

$w \Vdash_1 A \wedge B$ *iff* $w \Vdash_1 A$ *and* $w \Vdash_1 B$
$w \Vdash_0 A \wedge B$ *iff* $w \Vdash_0 A$ *or* $w \Vdash_0 B$

$w \Vdash_1 A \vee B$ *iff* $w \Vdash_1 A$ *or* $w \Vdash_1 B$
$w \Vdash_0 A \vee B$ *iff* $w \Vdash_0 A$ *and* $w \Vdash_0 B$

$w \Vdash_1 A \supset B$ *iff for all* $x \geq w$, $x \nVdash_1 A$ *or* $x \Vdash_1 B$
$w \Vdash_0 A \supset B$ *iff* $w \Vdash_1 A$ *and* $w \Vdash_0 B$

$w \Vdash_1 -A$ *iff* $w \Vdash_0 A$
$w \Vdash_0 -A$ *iff* $w \Vdash_1 A$

# Some facts about $N_3$

| | $N_3$ |
|---|---|
| $\models A \vee -A$ | invalid |
| $-(A \vee -A) \models B$ | valid |
| $\models -(A \wedge -A)$ | invalid |
| $(A \wedge -A) \models B$ | valid |
| $A \supset B \models -B \supset -A$ | invalid |
| $-(A \supset B) \models -(-B \supset -A)$ | valid |
| Double Negation Laws | valid |
| de Morgan's Laws | valid |